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A Coordinate-Free Approach to Wave Reflection from a Uniaxially Anisotropic Medium

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Abstract —This paper presents a coordinate-free method of solving the problem of electromagnetic wave reflection from the surface of a uniaxially anisotropic medium. Based on the direct manipulation of vectors, dyadics, and their invariants, the method eliminates the use of coordinate systems. It facilitates solutions and provides results in a greater generality. The paper contains the following results in coordinate-free forms: a) the dispersion equations; b) the directions of field vectors; c) the Poynting vectors (ray vectors) and group velocities; d) the laws of reflection and refraction; and e) the transmission and reflection coefficients. The results are valid for the incident wave having any polarization, and the optic axis of the uniaxial medium being arbitrarily oriented with respect to the interface and the plane of incidence.

I. Introduction

BECAUSE OF THE rapid advances in technology, wave propagation in anisotropic media such as plasmas, ferrites, etc., has become a subject of intense research [1]–[7]. The emergence of coherent light and optical fibers also makes wave propagation in dielectric crystals a topic of special interest [8]–[12].

In applied electromagnetics, the approach to solutions of various boundary value problems has been the coordinate method [8], [13]–[15]; that is, during the processes of solutions, one or more coordinate systems are introduced. For example, in considering wave propagation in an anisotropic crystal, we formulate and solve the problem with respect to a particular coordinate system—the principal coordinate system of the dielectric tensor [8]. However, when a boundary surface exists, the problem becomes more complex. In this case, two generally inconsistent requirements govern the choice of coordinate system. Inside the crystal, the principal coordinate is preferred, but on the boundary surface, a coordinate system with one of its coordinate planes coinciding with the surface is preferred. Using either system leads to a large number of simultaneous equations and ends in very cumbersome results [16]. Thus, only some special orientations of the optic axis with respect to the interface and the plane of incidence have been considered [17], [18].

In this paper, we shall present a coordinate-free method to solutions of wave reflection from a uniaxially anisotropic medium. We consider only the case when $\bar{\epsilon}$ is a tensor while μ is a scalar. The method applies equally well to the dual case of ferrites. Since the electric and magnetic fields are vector quantities, and they are related by the vector Maxwell equations and constitutive relations, we shall seek vector solutions directly from these vector equations. Based on the direct manipulation of vectors, dyadics, and their invariants, the method eliminates the use of

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coordinate systems. It facilitates solutions, condenses exposition, and provides results in a greater generality. It further renders physical concepts more tangible and easy to grasp [19].

II. Dispersion Equations and Directions of Field Vectors

For a monochromatic plane wave of the form

$$\mathcal{E} = \mathrm{Re}\{\boldsymbol{E}_0 \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\} \tag{1}$$

the Maxwell equations in a source-free region are

$$\omega\epsilon_0\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{E}_0 = -\boldsymbol{k}\times\boldsymbol{H}_0 \tag{2}$$

$$\omega\mu_0\boldsymbol{H}_0 = \boldsymbol{k}\times\boldsymbol{E}_0 \tag{3}$$

where $\bar{\boldsymbol{\epsilon}}$ is the dielectric tensor of the medium. With respect to the principal coordinate system the dielectric tensor of a uniaxial medium takes the matrix representation

$$\bar{\boldsymbol{\epsilon}} = \begin{bmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{bmatrix} \tag{4}$$

where two of the three diagonal elements are equal. We may also express $\bar{\boldsymbol{\epsilon}}$ in a coordinate-free dyadic form

$$\bar{\boldsymbol{\epsilon}} = \epsilon_\perp \bar{\boldsymbol{I}} + (\epsilon_\parallel - \epsilon_\perp)\hat{c}\hat{c} \tag{5}$$

where $\hat{c}$ is a unit vector in the direction of optic axis, and $\bar{\boldsymbol{I}}$ is the unit dyad. In crystal optics, (5) is associated with a uniaxial crystal [8]–[10]. In a magnetoplasma, it characterizes the medium when the externally applied constant magnetic field becomes very strong [1].

Making use of the Cayley–Hamilton theorem of matrix analysis [19], [20], we obtain, respectively, the determinant and adjoint of the sum of two 3 by 3 matrices

$$|\lambda\bar{\boldsymbol{I}} + \boldsymbol{ab}| = \lambda^2(\lambda + \boldsymbol{a}\cdot\boldsymbol{b}) \tag{6}$$

$$\mathrm{adj}(\lambda\bar{\boldsymbol{I}} + \boldsymbol{ab}) = \lambda[(\lambda + \boldsymbol{a}\cdot\boldsymbol{b})\bar{\boldsymbol{I}} - \boldsymbol{ab}]. \tag{7}$$

Hence

$$\bar{\boldsymbol{\epsilon}}^{-1} = \frac{1}{\epsilon_\perp}\bar{\boldsymbol{I}} + \left(\frac{1}{\epsilon_\parallel} - \frac{1}{\epsilon_\perp}\right)\hat{c}\hat{c}. \tag{8}$$

Eliminating $\boldsymbol{E}_0$ from (2) and (3), making use of (8), and noting from (3) that $\boldsymbol{k}\cdot\boldsymbol{H}_0 = 0$, we obtain

$$\bar{\boldsymbol{W}}(\boldsymbol{k})\cdot\boldsymbol{H}_0 = \boldsymbol{0} \tag{9}$$

where the wave matrix $\bar{\boldsymbol{W}}(\boldsymbol{k})$ of a uniaxial medium is given by

$$\bar{\boldsymbol{W}}(\boldsymbol{k}) = (k_0^2\epsilon_\perp - k^2)\bar{\boldsymbol{I}} - \frac{(\epsilon_\perp - \epsilon_\parallel)}{\epsilon_\parallel}(\boldsymbol{k}\times\hat{c})(\boldsymbol{k}\times\hat{c}). \tag{10}$$

Using the results of (6) and (7), we find the determinant and adjoint of the wave matrix

$$|\bar{\boldsymbol{W}}(\boldsymbol{k})| = (k_0^2\epsilon_\perp - k^2)^2(k_0^2\epsilon_\perp\epsilon_\parallel - \boldsymbol{k}\cdot\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{k})/\epsilon_\parallel \tag{11}$$

$$\mathrm{adj}\,\bar{\boldsymbol{W}}(\boldsymbol{k}) = \frac{(k_0^2\epsilon_\perp - k^2)}{\epsilon_\parallel} \cdot\left[(k_0^2\epsilon_\perp\epsilon_\parallel - \boldsymbol{k}\cdot\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{k})\bar{\boldsymbol{I}} + (\epsilon_\perp - \epsilon_\parallel)(\boldsymbol{k}\times\hat{c})(\boldsymbol{k}\times\hat{c})\right]. \tag{12}$$

A nonzero solution $\boldsymbol{H}_0$ of (9) exists provided that the determinant of the wave matrix vanishes. Hence, we obtain the dispersion equations

$$k^2 = k_0^2\epsilon_\perp \tag{13}$$

and

$$\boldsymbol{k}\cdot\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{k} = k_0^2\epsilon_\perp\epsilon_\parallel. \tag{14}$$

For a given direction of wave normal $\hat{\boldsymbol{k}}$, these two equations determine two values of wave numbers k. Since the wave number defined by (13) does not depend on the direction of wave normal $\hat{\boldsymbol{k}}$, as in the case of isotropic media, the corresponding wave is called the ordinary wave. Denoting this wave number by k_+, we have

$$k_+ = k_0\sqrt{\epsilon_\perp}. \tag{15}$$

On the other hand, the wave number defined by (14) does depend on the direction of wave normal $\hat{\boldsymbol{k}}$, and thus the corresponding wave is called the extraordinary wave. According to (14), the wave number of the extraordinary wave denoted by k_- is

$$k_- = k_0\left[\frac{\epsilon_\perp\epsilon_\parallel}{\epsilon_\perp(\hat{\boldsymbol{k}}\times\hat{c})^2 + \epsilon_\parallel(\hat{\boldsymbol{k}}\cdot\hat{c})^2}\right]^{1/2}. \tag{16}$$

Since multiplication of any solution of the homogeneous equation (9) by a constant yields another solution, (9) uniquely determines only the direction of $\boldsymbol{H}_0$, not its magnitude. For the ordinary wave, both the determinant and adjoint of the wave matrix vanish. In this case, the homogeneous equation (9) becomes

$$(\boldsymbol{k}_+\times\hat{c})\cdot\boldsymbol{H}_0 = 0.$$

Since $\boldsymbol{H}_0$ must also be perpendicular to $\boldsymbol{k}_+$, we thus choose the direction of $\boldsymbol{H}_0$ as

$$\boldsymbol{h}_+ = \frac{1}{\omega\mu_0}[\boldsymbol{k}_+\times(\boldsymbol{k}_+\times\hat{c})]. \tag{17}$$

The direction of the electric field vector $\boldsymbol{E}_0$ follows from the Maxwell equation

$$\boldsymbol{e}_+ = \boldsymbol{k}_+\times\hat{c}. \tag{18}$$

For the extraordinary wave, the adjoint of the wave matrix does not vanish. Hence, the direction of $\boldsymbol{H}_0$ is proportional to the columns of $\mathrm{adj}\,\bar{\boldsymbol{W}}(\boldsymbol{k}_-)$ [19], [20]

$$\boldsymbol{h}_- = [\mathrm{adj}\,\bar{\boldsymbol{W}}(\boldsymbol{k}_-)]\cdot\boldsymbol{u} \tag{19}$$

where $\boldsymbol{u}$ is an arbitrary vector. Using the results of (12), (16), and Maxwell's equations, we choose

$$\boldsymbol{h}_- = \omega\epsilon_0\epsilon_\perp(\boldsymbol{k}_-\times\hat{c}) \tag{20}$$

hence

$$\boldsymbol{e}_- = k_0^2\epsilon_\perp\hat{c} - (\boldsymbol{k}_-\cdot\hat{c})\boldsymbol{k}_-. \tag{21}$$

It is noted that for the ordinary wave, $\boldsymbol{e}_+$ is perpendicular to the plane formed by vectors $\boldsymbol{k}_+$ and optic axis $\hat{c}$ while $\boldsymbol{h}_+$ lies on the plane. On the other hand, for the extraordinary wave, vector $\boldsymbol{e}_-$ lies on the plane formed by vectors $\boldsymbol{k}_-$ and $\hat{c}$, but $\boldsymbol{h}_-$ is perpendicular to it.

III. Wave Vector Surfaces and Ray Vectors

We shall now consider some relations among the wave vector surface, Poynting's vector (or ray vector), velocity of energy transport, and group velocity in a uniaxially anisotropic medium. For an ordinary wave, the time average Poynting vector follows from (17) and (18)

$$\langle \boldsymbol{P}_+ \rangle = \frac{(\boldsymbol{k}_+ \times \hat{\boldsymbol{c}})^2}{2\omega\mu_0}\boldsymbol{k}_+ \tag{22}$$

since the total time average energy density is

$$\langle W_+ \rangle = \frac{\epsilon_0\epsilon_\perp}{2}(\boldsymbol{k}_+ \times \hat{\boldsymbol{c}})^2. \tag{23}$$

Hence, the velocity of energy transport becomes

$$\boldsymbol{v}_{E+} = \frac{\langle \boldsymbol{P}_+ \rangle}{\langle W_+ \rangle} = \frac{c}{\sqrt{\epsilon_\perp}}\hat{\boldsymbol{k}}_+. \tag{24}$$

A plot of dispersion equation as the wave normal $\hat{\boldsymbol{k}}$ takes all possible directions defines a wave vector surface. In the case of an ordinary wave, the wave vector surface defined by (15) is a surface of a sphere of radius $k_0\sqrt{\epsilon_\perp}$. The group velocity is thus

$$\boldsymbol{v}_{g+} = \frac{\partial\omega}{\partial\boldsymbol{k}} = \frac{c}{\sqrt{\epsilon_\perp}}\hat{\boldsymbol{k}}_+ \tag{25}$$

which is the same as the velocity of energy transport. From the definition of group velocity, we see that it is normal to the wave vector surface. Moreover, the dot product of the phase velocity

$$\boldsymbol{v}_{p+} = \frac{\omega}{k_+}\hat{\boldsymbol{k}}_+ = \frac{c}{\sqrt{\epsilon_\perp}}\hat{\boldsymbol{k}}_+ \tag{26}$$

and the group velocity gives

$$\boldsymbol{v}_{g+}\cdot\boldsymbol{v}_{p+} = \left(\frac{\omega}{k_+}\right)^2 = v_{p+}^2. \tag{27}$$

We may thus conclude that the ordinary wave of a uniaxial medium behaves as waves in isotropic media.

On the other hand, for the extraordinary wave, the wave vector surface defined by (16) may be written as

$$\frac{(\boldsymbol{k}_- \times \hat{\boldsymbol{c}})^2}{k_0^2\epsilon_\parallel} + \frac{(\boldsymbol{k}_-\cdot\hat{\boldsymbol{c}})^2}{k_0^2\epsilon_\perp} = 1 \tag{28}$$

which represents a surface of revolution about the optic axis $\hat{\boldsymbol{c}}$ and is an ellipsoid. The time average Poynting vector (or ray vector), from (20) and (21), is

$$\langle \boldsymbol{P}_- \rangle = \frac{1}{2}(\boldsymbol{e}_- \times \boldsymbol{h}_-) = \frac{\omega\epsilon_0\epsilon_\perp(\boldsymbol{k}_- \times \hat{\boldsymbol{c}})^2}{2\epsilon_\parallel}(\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_-). \tag{29}$$

The total time average energy density is

$$\langle W_- \rangle = \frac{k_0^2\epsilon_\perp^2\epsilon_0}{2}(\boldsymbol{k}_- \times \hat{\boldsymbol{c}})^2. \tag{30}$$

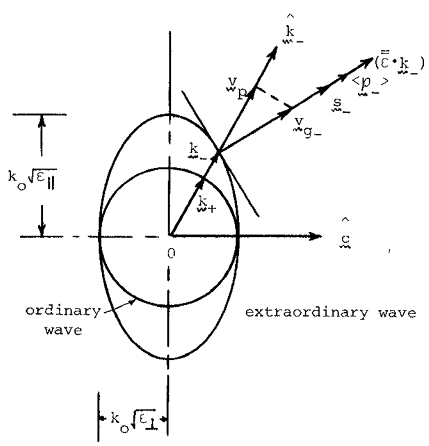

Fig. 1. Wave vector surfaces for a uniaxial medium when $\epsilon_\parallel > \epsilon_\perp$.

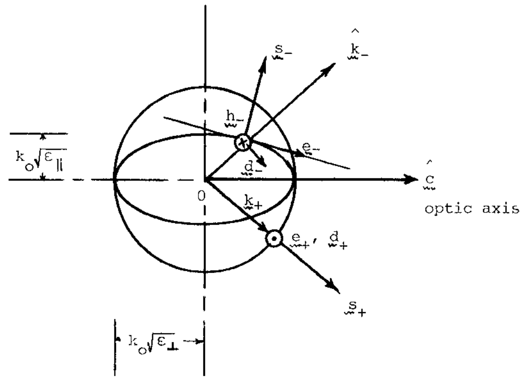

Fig. 2. Wave vector surfaces for a uniaxial medium when $\epsilon_\parallel < \epsilon_\perp$. Vectors $\boldsymbol{h}_-$ and $\boldsymbol{b}_-$ are directed into the paper. $\langle \boldsymbol{P}_- \rangle$ is normal to the wave vector surface at $\boldsymbol{k}_-$, $\boldsymbol{e}_-$ is perpendicular to $\langle \boldsymbol{P}_- \rangle$, and $\boldsymbol{d}_-$ is perpendicular to $\boldsymbol{k}_-$. Vectors $\boldsymbol{e}_+$ and $\boldsymbol{d}_+$ are directed out of the paper.

Hence, the velocity of energy transport

$$\boldsymbol{v}_{E-} = \frac{\langle \boldsymbol{P}_- \rangle}{\langle W_- \rangle} = \frac{c^2}{\omega\epsilon_\perp\epsilon_\parallel}(\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_-) \tag{31}$$

is again equal to the group velocity

$$\boldsymbol{v}_{g-} = \frac{\partial\omega}{\partial\boldsymbol{k}} = \frac{c^2}{\omega\epsilon_\perp\epsilon_\parallel}(\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{k}_-) \tag{32}$$

but is not equal to the phase velocity

$$\boldsymbol{v}_{p-} = \frac{\omega}{k_-}\hat{\boldsymbol{k}}_- = c\left[\frac{(\boldsymbol{k}_- \times \hat{\boldsymbol{c}})^2}{\epsilon_\parallel} + \frac{(\boldsymbol{k}_-\cdot\hat{\boldsymbol{c}})^2}{\epsilon_\perp}\right]^{1/2}\hat{\boldsymbol{k}}_-. \tag{33}$$

However, the dot product of group and phase velocities still yields

$$\boldsymbol{v}_{g-}\cdot\boldsymbol{v}_{p-} = \left(\frac{\omega}{k_-}\right)^2 = v_{p-}{}^2. \tag{34}$$

The above results can be illustrated graphically. Fig. 1 shows a cross-sectional view of a wave vector surface when $\epsilon_\parallel > \epsilon_\perp$ (for example, quartz). We see that in this case the ordinary wave travels faster than the extraordinary wave. Conversely, when $\epsilon_\parallel < \epsilon_\perp$ (for example, calcite) the ordinary wave in a uniaxially anisotropic medium travels slower than the extraordinary wave, as is shown in Fig. 2. In either case, we note that the direction of energy transport does not coincide with the direction of wave propagation $\hat{k}_-$, and the projection of the group velocity in the direction of wave normal gives the phase velocity of the wave.

IV. Laws of Reflection and Refraction

We shall next examine the problem of reflection and transmission of waves at the interface of an isotropic and uniaxial medium. The dispersion equations (13) and (14) cannot determine the wave vector $\boldsymbol{k}$ completely. However, at the interface of two media, the tangential component of $\boldsymbol{k}$ must be continuous, thus

$$\boldsymbol{k}_i \times \hat{\boldsymbol{q}} = \boldsymbol{k}_r \times \hat{\boldsymbol{q}} = \boldsymbol{k}_+ \times \hat{\boldsymbol{q}} = \boldsymbol{k}_- \times \hat{\boldsymbol{q}} \tag{35}$$

where $\boldsymbol{k}_i$ is the wave vector of the incident wave, $\hat{\boldsymbol{q}}$ is a unit vector normal to the interface pointing from isotropic medium I toward uniaxial medium II as shown in Fig. 3. $\boldsymbol{k}_r$, $\boldsymbol{k}_+$, and $\boldsymbol{k}_-$ are the wave vectors of the reflected and the two transmitted ordinary and extraordinary waves, respectively. Equation (35) is the vector forms of the laws of reflection and refraction. By introducing vectors

$$\boldsymbol{b} = \hat{\boldsymbol{q}} \times \boldsymbol{a} \tag{36}$$

and

$$\boldsymbol{a} = \boldsymbol{k}_i \times \hat{\boldsymbol{q}} = \boldsymbol{b} \times \hat{\boldsymbol{q}} \tag{37}$$

we may rewrite (35) as

$$\boldsymbol{k}_\alpha = \boldsymbol{b} + q_\alpha \hat{\boldsymbol{q}} \tag{38}$$

where the subscript α denotes i, r, $+$, or $-$. Equation (38) clearly shows that for a fixed origin 0 on the interface, the tips of all the wave vectors drawn from 0 must lie on a straight line that passes through the tip of vector $\boldsymbol{b}$ and is parallel to $\hat{\boldsymbol{q}}$ (see Fig. 3). To determine the wave vector $\boldsymbol{k}_+ = \boldsymbol{b} + q_+\hat{\boldsymbol{q}}$ of the transmitted ordinary wave in the uniaxial medium II, we substitute $\boldsymbol{k}_+$ into the dispersion equation (13) and obtain

$$q_+ = +\left(k_0^2\epsilon_\perp - a^2\right)^{1/2}. \tag{39}$$

The choice of the positive sign in front of the square root is dictated by the fact that the energy carried by the transmitted wave should flow toward medium II, that is, $\boldsymbol{k}_+ \cdot \hat{\boldsymbol{q}} = q_+ > 0$. Similarly, substituting the wave vector $\boldsymbol{k}_- = \boldsymbol{b} + q_-\hat{\boldsymbol{q}}$ of the transmitted extraordinary wave into the dispersion equation (14), we obtain

$$(\hat{\boldsymbol{q}}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}})q_-^2 + 2(\boldsymbol{b}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}})q_- + (\boldsymbol{b}\cdot\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{b}) - k_0^2\epsilon_\perp\epsilon_\parallel = 0. \tag{40}$$

This is a quadratic equation in q_-. Again we must choose solution q_- so that $\langle \boldsymbol{P}_- \rangle \cdot \hat{\boldsymbol{q}} > 0$. From (29) we see that this condition corresponds to $\boldsymbol{k}_-\cdot\bar{\boldsymbol{\epsilon}}\cdot\boldsymbol{q} > 0$, or

$$q_- > -\frac{\boldsymbol{b}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}}}{\hat{\boldsymbol{q}}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}}}. \tag{41}$$

From (40) to (41) we may now obtain

$$q_- = \frac{-\boldsymbol{b}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}} + \zeta}{\hat{\boldsymbol{q}}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}}} \tag{42}$$

where

$$\zeta = \left[k_0^2\epsilon_\perp\epsilon_\parallel(\hat{\boldsymbol{q}}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}}) - \boldsymbol{a}\cdot(\operatorname{adj}\bar{\boldsymbol{\epsilon}})\cdot\boldsymbol{a}\right]^{1/2}. \tag{43}$$

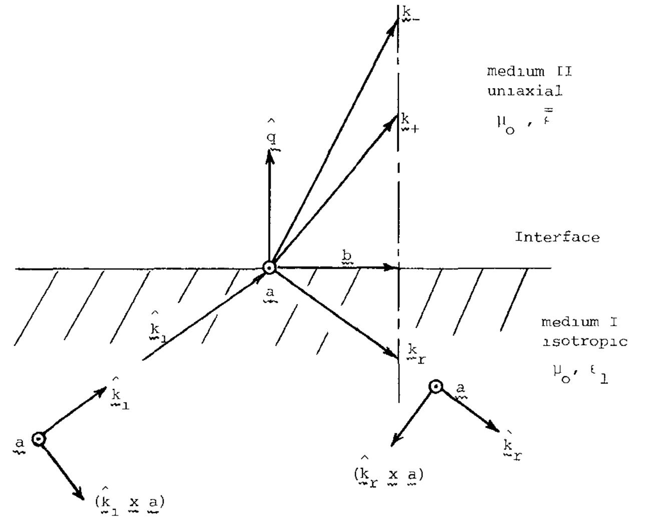

Fig. 3. Geometry of the reflection problem and orientations of wave vectors.

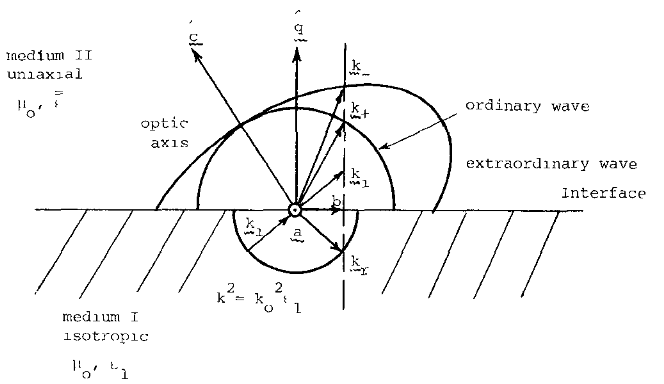

Fig. 4. Geometrical determination of wave vectors at the interface of an isotropic and a uniaxial medium. Vector $\boldsymbol{a}$ is directed out of the paper.

Hence

$$\boldsymbol{k}_- = \frac{(\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}})\times\boldsymbol{a} + \zeta\hat{\boldsymbol{q}}}{\hat{\boldsymbol{q}}\cdot\bar{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{q}}}. \tag{44}$$

A geometrical determination of wave vectors at the interface of an isotropic and uniaxial medium is shown in Fig. 4.

V. Transmission and Reflection Coefficient Matrices

Knowing the wave vectors of all the waves, we may now proceed to find the reflection and transmission coefficients at the interface of an isotropic–uniaxial medium. We decompose the amplitude vectors of the incident and the reflected waves in isotropic medium I into components perpendicular and parallel to the plane of incidence (see Fig. 3)

$$\boldsymbol{E}_{0i} = A_\perp \boldsymbol{a} + A_\parallel(\hat{\boldsymbol{k}}_i \times \boldsymbol{a}) \tag{45}$$

$$\boldsymbol{H}_{0i} = (\epsilon_0\epsilon_1/\mu_0)^{1/2}\left[A_\perp(\hat{\boldsymbol{k}}_i\times\boldsymbol{a}) - A_\parallel\boldsymbol{a}\right] \tag{46}$$

and

$$\boldsymbol{E}_{0r} = B_\perp \boldsymbol{a} + B_\parallel(\hat{\boldsymbol{k}}_r \times \boldsymbol{a}) \tag{47}$$

$$\boldsymbol{H}_{0r} = (\epsilon_0\epsilon_1/\mu_0)^{1/2}\left[B_\perp(\hat{\boldsymbol{k}}_r\times\boldsymbol{a}) - B_\parallel\boldsymbol{a}\right] \tag{48}$$

where ϵ_1 is the dielectric constant of the isotropic medium I. The amplitude vectors of two transmitted waves in uniaxial medium II are as follows. For the ordinary wave

$$E_{0+} = C_+ e_+ \quad H_{0+} = C_+ h_+ \tag{49}$$

where e_+ and h_+ are given by (18) and (17), respectively. For the extraordinary wave

$$E_{0-} = C_- e_- \quad H_{0-} = C_- h_- \tag{50}$$

where e_- and h_- are given by (21) and (20), respectively. Matching the boundary conditions at the interface, we find the amplitudes C_+ and C_- of the transmitted waves in terms of the known amplitudes $A_\perp$ and $A_\parallel$ of the incident wave

$$\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} A_\perp \\ A_\parallel \end{bmatrix} \tag{51}$$

where the transmission coefficients are given by

$$\begin{aligned} T_{11} &= M_{11}/\Delta \quad T_{12} = M_{12}/\Delta \\ T_{21} &= M_{21}/\Delta \quad T_{22} = M_{22}/\Delta \end{aligned} \tag{52}$$

and

$$\begin{aligned} M_{11} &= -2q_t a^2(X+Y)/[k_+\cdot(a\times\hat{c})] \\ M_{21} &= 2q_t a^2(U+Z)/k_0^2\epsilon_\perp(a\cdot\hat{c}) \\ &= 2q_t a^2(k_t^2 q_+ + k_0^2\epsilon_\perp q_t)(a\cdot\hat{c}) \\ M_{12} &= 2k_0^2 k_t a^2\epsilon_\perp q_t(q_t+q_-)(a\cdot\hat{c}) \\ M_{22} &= 2k_t a^2 q_t(q_t+q_+)[k_+\cdot(a\times\hat{c})] \end{aligned} \tag{53}$$

$$\begin{aligned} \Delta &= (q_i+q_+)(X+Y)+(q_i+q_-)(U+Z) \\ X &= k_0^2 q_t\epsilon_\perp[k_+\cdot(a\times\hat{c})][k_-\cdot(a\times\hat{c})] \\ Y &= k_t^2[k_+\cdot(a\times\hat{c})][q_+^2(b\cdot\hat{c})-q_-a^2(\hat{q}\cdot\hat{c})] \\ U &= k_0^2k_t^2\epsilon_\perp q_+(a\cdot\hat{c})^2 \\ Z &= k_0^4\epsilon_\perp^2 q_i(a\cdot\hat{c})^2. \end{aligned} \tag{54}$$

Similarly, we obtain the amplitudes $B_\perp$ and $B_\parallel$ of the reflected wave in terms of the known amplitudes $A_\perp$ and $A_\parallel$ of the incident wave

$$\begin{bmatrix} B_\perp \\ B_\parallel \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} A_\perp \\ A_\parallel \end{bmatrix} \tag{55}$$

where the reflection coefficients are given by

$$\begin{aligned} \Gamma_{11} &= [(q_i-q_+)(X+Y)+(q_i-q_-)(U+Z)]/\Delta \\ \Gamma_{12} &= 2(q_+-q_-)(V-L)/\Delta \\ \Gamma_{21} &= 2(q_- - q_+)(V+L)/\Delta \\ \Gamma_{22} &= [(q_t+q_+)(X-Y)+(q_t+q_-)(Z-U)]/\Delta \end{aligned} \tag{56}$$

and

$$\begin{aligned} V &= k_0^2\epsilon_\perp k_t q_t q_+(a\cdot\hat{c})(b\cdot\hat{c}) \\ L &= k_0^2\epsilon_\perp k_t q_t a^2(a\cdot\hat{c})(\hat{q}\cdot\hat{c}) \end{aligned} \tag{57}$$

$$\begin{aligned} V-L &= k_0^2\epsilon_\perp k_t q_t(a\cdot\hat{c})[k_+\cdot(a\times\hat{c})] \\ V+L &= k_0^2\epsilon_\perp k_i q_t(a\cdot\hat{c})[q_+(b\cdot\hat{c})+a^2(\hat{q}\cdot\hat{c})]. \end{aligned} \tag{58}$$

Equations (51) and (55) give, respectively, the transmission and reflection coefficients of a wave that is incident from an isotropic to a uniaxial medium. The results are obtained in the most general vector forms with an incident wave of any polarization, and the optic axis $\hat{c}$ arbitrarily oriented with respect to the interface and the plane of incidence.

An examination of (54) shows that when the direction of the optic axis $\hat{c}$ is reversed, the reflection coefficients given in (56) and the reflected field vectors in (47) and (48) remain unchanged. On the other hand, from (53) we see that the transmission coefficients given in (52) change signs when we reverse the direction of $\hat{c}$. However, according to (17), (18), (20), and (21), the field vectors E_{0+}, H_{0+}, E_{0-}, and H_{0-} of the transmitted wave again remain unchanged.

It is interesting to note that in the presence of a uniaxial medium II, the reflection and transmission coefficient matrices contain both diagonal and off-diagonal elements. Thus, for a linearly polarized incident wave with the electric field intensity perpendicular to the plane of incidence, the electric vector of the reflected wave will have components both perpendicular and parallel to the plane of incidence.

In the special case of medium II being isotropic, that is, $\epsilon_\perp = \epsilon_\parallel = \epsilon_2$, we have

$$\begin{aligned} q_+ &= q_- = (k_0^2\epsilon_2 - a^2)^{1/2} = q_t \\ k_+^2 &= k_-^2 = k_0^2\epsilon_2 = k_t^2 \end{aligned} \tag{59}$$

and

$$k_+ = k_- = b + q_t\hat{q} = k_t. \tag{60}$$

Furthermore

$$E_{0-} = -C_+[k_t\times(k_t\times\hat{c})]. \tag{61}$$

Since $\hat{c}$ may now be in any direction, and in order to identify E_{0+} and E_{0-} with the components perpendicular and parallel to the plane of incidence, we choose $\hat{c} = \hat{q}$; thus

$$\begin{aligned} E_{0+} &= C_+ a \\ E_{0-} &= -C_-(k_t\times a). \end{aligned} \tag{62}$$

Substitution of the above into (52)–(58) yields

$$\frac{C_+}{A_\perp} = \frac{2q_i}{q_t+q_t} \quad \frac{C_-}{A_\parallel} = \frac{-2k_t q_t}{k_t^2 q_i + k_t^2 q_t}$$
$$T_{12} = T_{21} = 0 \tag{63}$$

and

$$\Gamma_{11} = \frac{q_t - q_t}{q_i + q_t} \quad \Gamma_{22} = \frac{k_t^2 q_t - k_t^2 q_t}{k_t^2 q_t + k_t^2 q_t}$$
$$\Gamma_{12} = \Gamma_{21} = 0. \tag{64}$$

These are the well-known Fresnel's formulas for two isotropic media [14]. In this case, we see that all the off-diagonal terms of the transmission and reflection coefficient matrices vanish. Thus, we may treat the perpendicular and parallel polarizations separately.

VI. Normal Incidence

In the case of normal incidence, formulas (52)–(58) are no longer valid because the concept of the plane of incidence loses its meaning. In this case, the wave vectors take the form

$$\boldsymbol{k}_i = k_i \hat{\boldsymbol{q}} = -\boldsymbol{k}_r$$
$$\boldsymbol{k}_+ = k_+ \hat{\boldsymbol{q}} \quad \boldsymbol{k}_- = k_- \hat{\boldsymbol{q}} \tag{65}$$

where

$$k_i = k_0 \sqrt{\epsilon_1} \quad k_+ = k_0 \sqrt{\epsilon_\perp}$$
$$k_- = k_0 \left(\frac{\epsilon_\perp \epsilon_\|}{\hat{\boldsymbol{q}} \cdot \bar{\epsilon} \cdot \hat{\boldsymbol{q}}} \right)^{1/2}. \tag{66}$$

We now consider the plane formed by vectors $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{q}}$ as though it were the plane of incidence. Henceforth, the subscripts $\perp$ and $\|$ will be used in this sense. Decomposing the field vectors of the incident and the reflected waves into components perpendicular and parallel to the plane formed by vectors $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{q}}$, we obtain the following.

For the incident wave

$$\boldsymbol{E}_{0i} = A_\perp (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}}) + A_\| [\hat{\boldsymbol{q}} \times (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})]$$
$$\boldsymbol{H}_{0i} = \frac{1}{\omega \mu_0} (\boldsymbol{k}_i \times \boldsymbol{E}_{0i})$$
$$= (\epsilon_0 \epsilon_1 / \mu_0)^{1/2} \{ A_\perp [\hat{\boldsymbol{q}} \times (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})] - A_\| (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}}) \}. \tag{67}$$

For the reflected wave

$$\boldsymbol{E}_{0r} = B_\perp (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}}) + B_\| [\hat{\boldsymbol{q}} \times (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})]$$
$$\boldsymbol{H}_{0r} = (\epsilon_0 \epsilon_1 / \mu_0)^{1/2} \{ -B_\perp [\hat{\boldsymbol{q}} \times (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})] + B_\| (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}}) \}. \tag{68}$$

For the transmitted ordinary wave

$$\boldsymbol{E}_{0+} = C_+ (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})$$
$$\boldsymbol{H}_{0+} = (\epsilon_0 \epsilon_\perp \mu_0)^{1/2} C_+ [\hat{\boldsymbol{q}} \times (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})]. \tag{69}$$

For the transmitted extraordinary wave

$$\boldsymbol{E}_{0-} = C_- \left[k_0^2 \epsilon_\perp \hat{\boldsymbol{c}} - k_-^2 (\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{c}}) \hat{\boldsymbol{q}} \right]$$
$$\boldsymbol{H}_{0-} = \omega \epsilon_0 \epsilon_\perp k_- C_- (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}}). \tag{70}$$

Matching the boundary conditions at the interface and noting that $(\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})$ and $[\hat{\boldsymbol{q}} \times (\hat{\boldsymbol{q}} \times \hat{\boldsymbol{c}})]$ are two linearly independent vectors, we obtain

$$\frac{C_+}{A_\perp} = \frac{2k_i}{k_i + k_+} \quad \frac{C_-}{A_\|} = \frac{-2k_i}{k_0^2 \epsilon_\perp (k_i + k_-)} \tag{71}$$

and

$$\frac{B_\perp}{A_\perp} = \frac{k_i - k_+}{k_i + k_+} \quad \frac{B_\|}{A_\|} = \frac{k_i - k_-}{k_i + k_-}. \tag{72}$$

Relations (52) and (56) together with (71) and (72) completely solve the problem of wave reflection from a uniaxially anisotropic medium.

VII. Conclusion

In this paper we presented a coordinate-free method to solve the general problem of electromagnetic wave reflection from the surface of a uniaxially anisotropic medium. The results are expressed in coordinate-free forms. They have the advantages that the incident wave can have any polarization, and that the optic axis of the uniaxial medium can be arbitrarily oriented with respect to the interface and the plane of incidence.

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